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Object of the Paper.

In the quantitative discussion of the behavior of an alternating-current electric filter under variable impressed frequency, it has commonly been assumed[1] that the load at the receiving end has an impedance equal to the characteristic or surge impedance of the filter; in other words that it behaves as though the load at the receiving end was an indefinite prolongation of the same filter. It is proposed to consider a case of a simple uniform, low-pass, five-section filter, with a constant load at the receiving end, and to indicate that in the case of such a relatively high degree of complexity, the behavior of the filter can be readily computed on the principle of simple alternating-current artificial lines.

The results have been obtained by calculation only, as the assumed filter was not actually tested or constructed. A sufficient number of observations have, however, been collected upon actual filters of the same type, to justify reporting the computations. As is customary in such calculations, the filter has been assumed as operating without internal losses or dissipation of power. If, however, definite small uniform resistance and dielectric losses had been assumed, corresponding to those which occur in filters actually employed, the computations would not thereby have been made much more difficult. While they would have taken a little more time to work out, the difference in such a case as this would not be great. The method here indicated is recommended as of general and convenient application.

Particulars Concerning the Filter Selected for Computation.

Fig. 1 represents the T section selected. It has an inductance of $\mathfrak{L} = 0.040$ henry in the series element AOB, and a capacitance of $C = 10^{-6}$ farad, or 1 microfarad, in shunt. The resistance of the inductor is taken as negligible, and likewise the leakance of the condenser.

Fig. 2 represents the series connection of 5 such identical T filter sections. The filter is thus assumed to contain a total series induct-

[1] One case has however already been briefly reported by F. S. Dellenbaugh, using the method described in this paper. See Bibliography 10, p. 21, Fig. 11.

ance of 0.2 henry, and a total shunt capacitance of 5 microfarads. The load σ between the receiving-end terminal B and the ground is a constant resistance of 183.3 ohms, either alone, or in series with a constant inductance of 0.0265 henry, according to the position of the switch S.

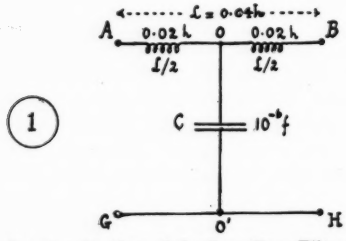

FIG. 1. Section of the Low-Pass Filter Selected.

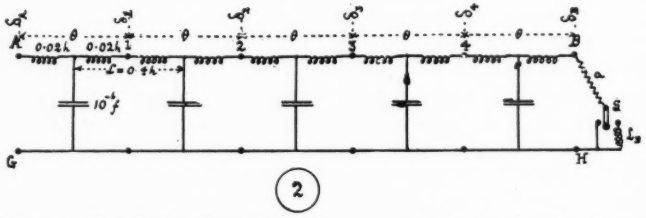

FIG. 2. Five-Section Low-Pass Filter corresponding to the Constants of Fig. 1 with a Load σ at the Receiving end B.

It is well known that, at a given impressed frequency, the behavior of any symmetrical filter depends on two and only two quantities connected therewith; namely, (1) the hyperbolic angle θ of the filter and (2) its surge impedance z_0 ohms $\angle$. Both θ and z_0 vary with the frequency. Moreover, in the case of a filter like that of Fig. 2, containing n identical symmetrical sections, each section will have the same angle θ, and surge impedance z_0 so that

$$\Theta = n\theta \qquad \begin{matrix}\text{hyperbolic} \\ \text{radians or hyps } \angle\end{matrix} \qquad (1)$$

According to the rule[2] for determining the angular velocity of cut off in a filter like that of Figs. 1 and 2, if $\mathcal{L}$ is the series inductance

[2] Bibliography, 2, 6.

in henrys per section, and C is the shunt capacitance in farads per section,

$$\omega_0 = \frac{2}{\sqrt{\mathcal{L}C}} = 2\pi f_0 \qquad \frac{\text{radians}}{\text{sec.}} \qquad (2)$$

f_0 being the cut-off frequency.

If the angular velocity impressed on the filter is ω, we may call the ratio of this to the cut-off angular velocity, the frequency ratio u, or

$$u = \frac{\omega}{\omega_0} = \frac{\omega\sqrt{\mathcal{L}C}}{2} = \pi f\sqrt{\mathcal{L}C} = \frac{f}{f_0} \qquad \text{numeric} \qquad (3)$$

This type of filter allows alternating currents to pass for values of the frequency ratio is less than unity, and arrests them more or less completely when this ratio exceeds unity.

At any impressed frequency ratio ω, the section angle of this type of filter is defined by the relation

$$\sinh\frac{\theta}{2} = ju \qquad \text{numeric } \angle \qquad (4)$$

or

$$\theta = 2\sinh^{-1}(ju) \qquad \text{hyps } \angle \qquad (5)$$

Fig. 3 shows a portion of a chart[3] for obtaining antihyperbolic sines, or $\sinh^{-1}x$. At A is the origin of coördinates. Entering the chart on the rectangular coördinate network, we start from A and move up along the j line ABC, until we reach the value chosen for u. Holding this point with a blunt needle, we read off its corresponding value on the curvilinear coördinates, which are of the type $x + jq$. The real component x is in hyperbolic radians. The j component q is in circular quadrants of arc, each quadrant being 90°.

On page 465 is a brief Table of $\sinh^{-1}(ju)$, as obtainable from the chart Fig. 3, but to a lower degree of precision.

It will be seen that as u increases from 0 to 1.0, θ increases from $0 + j0$ to $0 + j\underline{2}$ in quadrant measure, or to $0 + j\pi$ in circular measure. This means that θ is entirely imaginary, or has no real component, as far as $u = 1.0$. Immediately on passing $u = 1$, however, the value of θ undergoes no further change in the imaginary part $j\theta_2$; but rapidly develops a real part θ_1. An artificial line section of angle θ, forming part of an indefinitely long line, subjects any voltage or current which traverses it to an attenuation $\varepsilon^{-\theta} = \varepsilon^{-(\theta_1+j\theta_2)} = \varepsilon^{-\theta_1} \sphericalangle \theta_2$, where $\varepsilon = 2.718\ldots$ the Napierian base.

[3] Bibliography 4.

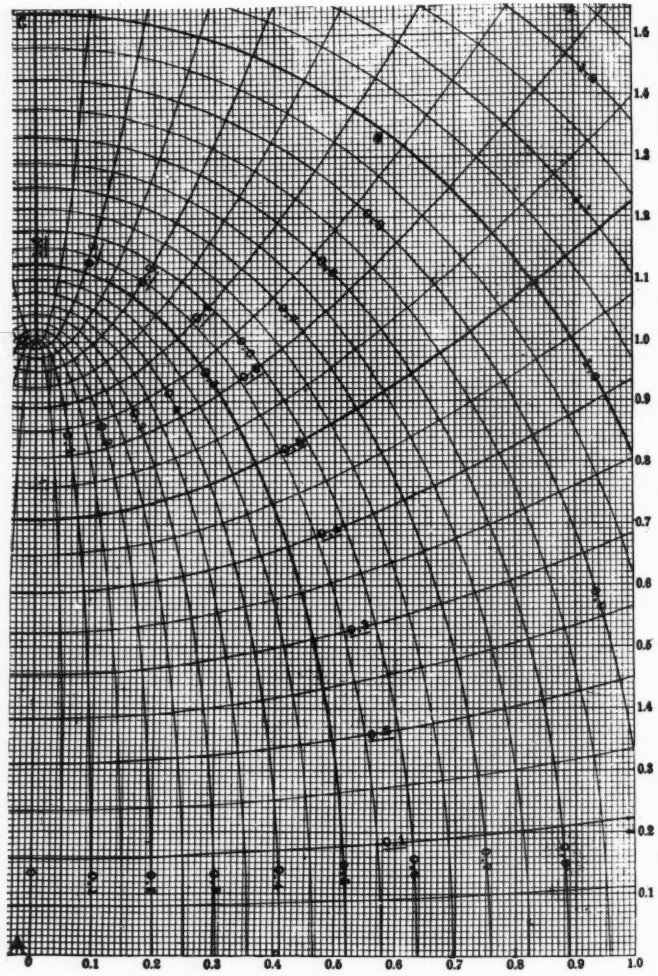

FIG. 3. Portion of Chart for evaluating $\sinh^{-1} jy = \frac{\theta_1 + j\theta_2}{2}$.

TABLE I

SECTION ANGLE θ FOR A T-FILTER OF THE TYPE SHOWN IN FIG. 1 AS A FUNCTION OF THE FREQUENCY RATIO u.

ω	$ju = \frac{j\omega}{\omega_0}$	$\frac{\theta}{2} = \sinh^{-1}(ju)$		$\theta = \theta_1 + j\theta_2$	
		quadrants	cir. radians	quadrants	cir. radians
		$\frac{\theta_1}{2} + j\frac{q}{2}$	$\frac{\theta_1}{2} + j\frac{\theta_2}{2}$	$\theta_1 + jq$	$\theta_1 + j\theta_2$
0	0	$0 + j\,0$	$0 + j\,0$	$0 + j\,0$	$0 + j\,0$
1000	$j\,0.1$	$0 + j\,0.0638$	$0 + j\,0.1002$	$0 + j\,0.1276$	$0 + j\,0.2004$
2000	$j\,0.2$	$0 + j\,0.1283$	$0 + j\,0.2014$	$0 + j\,0.2566$	$0 + j\,0.4028$
3000	$j\,0.3$	$0 + j\,0.194$	$0 + j\,0.3047$	$0 + j\,0.388$	$0 + j\,0.6094$
4000	$j\,0.4$	$0 + j\,0.262$	$0 + j\,0.4115$	$0 + j\,0.524$	$0 + j\,0.8230$
5000	$j\,0.5$	$0 + j\,0.333$	$0 + j\,0.5236$	$0 + j\,0.666$	$0 + j\,1.0472$
6000	$j\,0.6$	$0 + j\,0.4095$	$0 + j\,0.6435$	$0 + j\,0.819$	$0 + j\,1.2870$
7000	$j\,0.7$	$0 + j\,0.493$	$0 + j\,0.7753$	$0 + j\,0.986$	$0 + j\,1.5506$
8000	$j\,0.8$	$0 + j\,0.590$	$0 + j\,0.9273$	$0 + j\,1.180$	$0 + j\,1.8546$
8100	$j\,0.81$	$0 + j\,0.601$	$0 + j\,0.9442$	$0 + j\,1.202$	$0 + j\,1.8884$
9000	$j\,0.9$	$0 + j\,0.712$	$0 + j\,1.1197$	$0 + j\,1.424$	$0 + j\,2.2394$
10000	$j\,1.0$	$0 + j\,1.0$	$0 + j\,1.5708$	$0 + j\,2.0$	$0 + j\,3.1416$
11000	$j\,1.1$	$0.4436 + j\,1.0$	$0.4436 + j\,1.5708$	$0.8872 + j\,2.0$	$0.8872 + j\,3.1416$
12000	$j\,1.2$	$0.6223 + j\,1.0$	$0.6223 + j\,1.5708$	$1.2446 + j\,2.0$	$1.2446 + j\,3.1416$
13000	$j\,1.3$	$0.7565 + j\,1.0$	$0.7565 + j\,1.5708$	$1.513 + j\,2.0$	$1.513 + j\,3.1416$
14000	$j\,1.4$	$0.8670 + j\,1.0$	$0.8670 + j\,1.5708$	$1.734 + j\,2.0$	$1.734 + j\,3.1416$
15000	$j\,1.5$	$0.9624 + j\,1.0$	$0.9624 + j\,1.5708$	$1.9248 + j\,2.0$	$1.9248 + j\,3.1416$

Consequently if θ_1 vanishes, $\varepsilon^{-\theta} = \varepsilon^{-j\theta_2} = 1 \angle \theta_2$ a mere versor, or phase-shifting operator, causing the voltage or current at its receiving-end terminals to lag in phase θ_2 circular radians with respect to the corresponding quantity at sending-end terminals; without altering the magnitude or size of the traversing quantity. When, however, there is a real component θ_1, not only is there slope attenuation (in phase) but there is also size attenuation (in magnitude). At $u = 1.5$, for example, $\theta = 1.9248 + j\underline{2}$ and $\varepsilon^{-\theta} = \varepsilon^{-1.9248-j\pi} = 0.146 \angle 180°$. The voltage or the current will fall in this case to 14.6 per cent in size, and by 180° in slope, when passing through the filter section, provided that there are a large number of similar sections beyond.

In the case of the particular values of $\mathfrak{L}$ and C shown in the T section of Fig. 1, the cut-off frequency $\omega_0 = 10{,}000$ radians per second, ($f_0 = 1591 \cong$) by (2). The values of impressed angular velocity are then given in the last column of Table I.

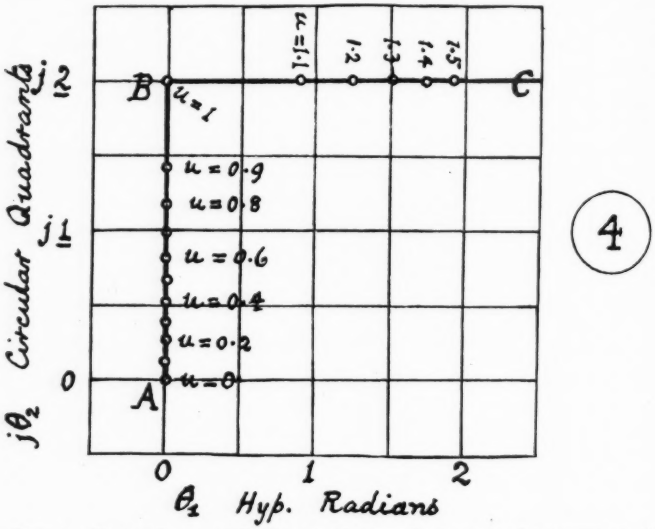

FIG. 4. Vector Locus of Section Angle θ as the impressed frequency ratio u is varied from 0 to 1.5.

Fig. 4 is a vector diagram of the section angle θ for the filter Fig. 1, as a function of impressed frequency ratio u. The vector locus is a pair of perpendicular straight lines, AB and BC. The turning point

at $u = 1$ corresponds to the cut off angular velocity ω_0. As u is increased from 1.5 to infinity, the horizontal locus BC advances continuously toward the right hand.

The rate of increase of section angle θ with respect to a change in u, is expressed by

$$d\theta = j\frac{2du}{\sqrt{1-u^2}} = j\frac{2du}{\cosh\left(\frac{\theta}{2}\right)} = j\frac{2du}{\cos\left(\frac{\theta_2}{2}\right)} \quad \text{hyps } \angle \quad (6)$$

below the cut-off value $u = 1$ and

$$d\theta = \frac{2du}{\sqrt{u^2-1}} = \frac{2du}{\cosh\left(\frac{\theta}{2}\right)} = \frac{2du}{\sinh\left(\frac{\theta_1}{2}\right)} \quad \text{hyps} \quad (7)$$

beyond the cut-off. Thus, at $u = 0.8$, or $\omega = 8000$ in the case considered, $d\theta = j2\,(du/0.6) = j(du/0.3)$. That is for a small change in du amounting say to 0.01, or 100 in ω, $d\theta = j\,0.033$ hyps; i.e. 0.033 circular radians or 0.0212 quadrant. This is in substantial agreement with the entry in Table I for $u = 0.81$.

At $u_0 = 1$, or the cut-off frequency, $d\theta/du$ becomes infinite, which would mean that an indefinitely small change in u would produce an indefinitely large change in θ at this point; but this refers only to a perfect filter devoid of all losses. In the presence of any appreciable loss, the rate of change $d\theta/du$ becomes finite, but may be large. The larger it is, the better the cut-off behavior of the filter.

Angle of Multiple-Section Filter.

Fig. 5 gives the vector locus diagram ABC of the angle Θ subtended by the entire filter of Fig. 2. By comparison with Fig. 4, it will be observed that the effect of the five sections is to make the rate of cut-off five times as great as with a single section. This is known to be a characteristic property of multiple-section filters. That is in changing u from 1.0 to 1.1, the increase in θ is 0.89 hyp. in Fig. 4, and 4.45 in Fig. 5.

Surge Impedance of One Section or of the Five-Section Filter.

By the regular rule for determining the surge impedance of a T section as in Fig. 1,

$$z_0 = \sqrt{\frac{\mathcal{L}}{C}}\cosh\frac{\theta}{2} = z_{00}\cosh\frac{\theta}{2} \qquad \text{ohms } \angle \quad (8)$$

or in the case considered, 200 cosh $\theta/2$, where θ is the section angle. The surge impedance thus varies with the impressed frequency.

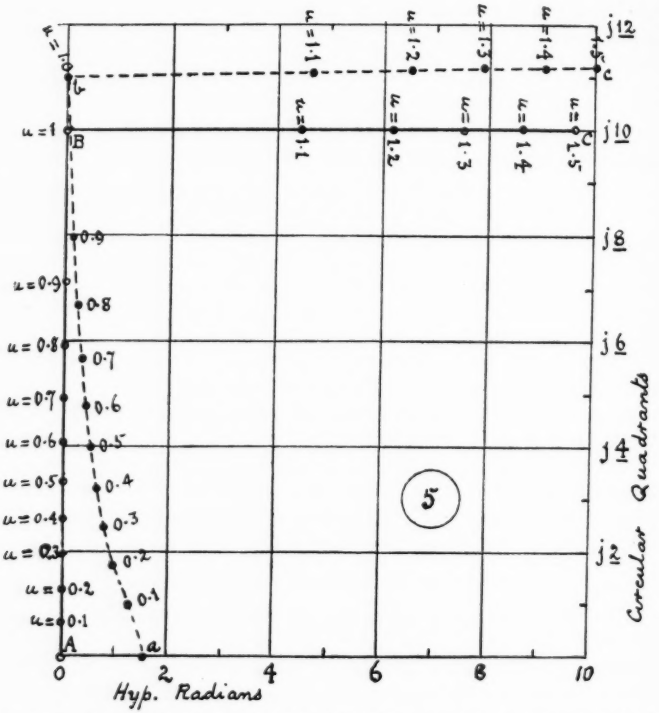

FIG. 5. Vector Locus ABC of the total filter angle Θ as u varies from 0 to 1.5; also the vector locus abc of the position angle δ_A when the load $\sigma = 183.3$ ohms + 0.0265 h.

Fig. 6 shows the vector locus. As ω increases from 0 to $\omega_0 = 10{,}000$, $(u = 1)$, z_0 moves along the graph AB following the resistance axis from 200 ohms to 0. At the cut-off point, z_0 vanishes, for the case assumed of a lossless filter. From and beyond ω_0, z_0 becomes a pure inductive reactance or $+ j$ quantity, along the branch BC, increasing rapidly at first, and afterwards more slowly.

Position Angle δ_B at the Receiving End of the Filter under Load.

The position angle δ_B (Fig. 2) of the receiving end B of the filter line Fig. 2, under a given load σ ohms $\angle$ and under an impressed angular velocity ω, is equal to the angle θ' of that load, and is found from the well known relation

$$\delta_B = \theta' = \theta_1' + j\theta_2' = \tanh^{-1}\frac{\sigma}{z_0} \qquad \text{hyps } \angle \quad (9)$$

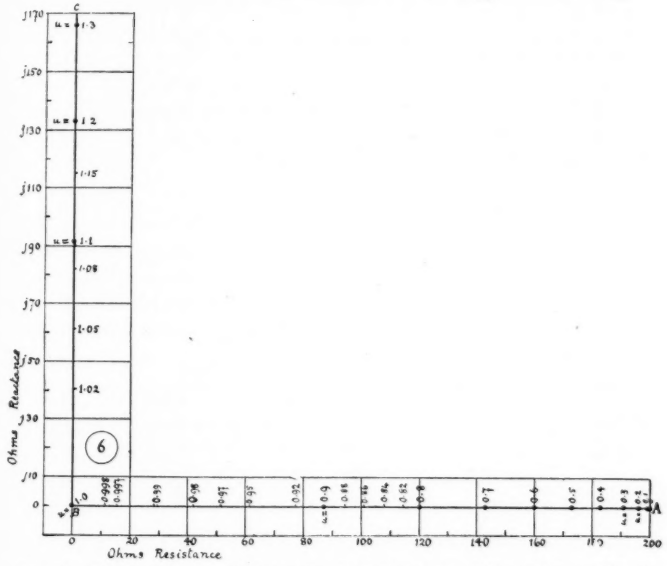

Fig. 6. Vector Locus of the surge impedance z_0 of the filter as the frequency ratio u is varied from 0 to 1.3 or ω from 0 to 13000.

But the impedances σ and z_0 will, in general, vary with the angular velocity ω; so that the angle θ' also varies with ω. The antitangent of the complex ratio σ/z_0 is found from a chart of tanh and $\tanh^{-1}$. Fig. 7 shows four vector loci of δ_B at varying angular velocities, for different loads σ connected to the receiving-end B of the filter in Fig. 2. The absscisas of Fig. 7 are in real hyperbolic radians θ_1'. The ordinates are in circular quadrants of arc $(j\theta_2')$. Thus $j\,\underline{1.0}$ represents one quadrant, ninety degrees, or $\pi/2$ radians. The four loads σ whose angles θ' are plotted in Fig. 7 as a function of u are:

(1) A pure resistance load $\sigma = 183$ ohms, graph $A\ b'\ d'\ e'\ D\ f$.
(2) A resistance of 183.3 ohms in series with a condenser of 2.65 microfarads, graph $D\ F\ G\ H\ D\ J\ K$.
(3) A resistance of 183.3 ohms in series with an inductance of 26.5 millihenries, graph $A\ B\ C\ D\ E$.
(4) A resistance of 156 ohms in series with an inductance of 22.5 millihenries, graph $a\ b\ c\ D\ e$.

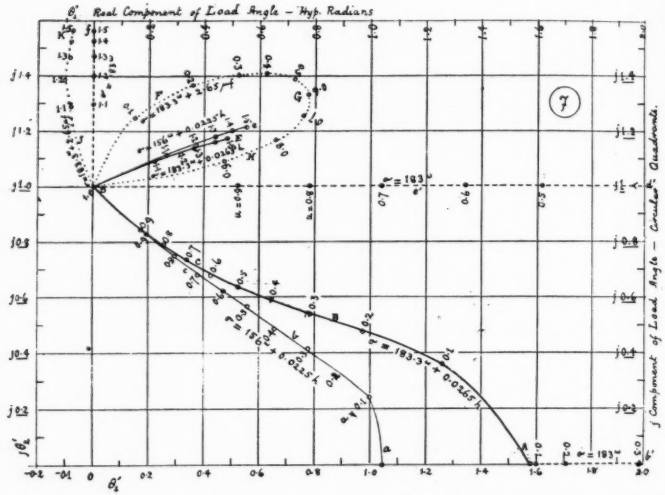

FIG. 7. Vector Loci of δ_B or of angles subtended by four different loads σ connected successively to the receiving end B of the filter shown in Fig. 2.

Different values of u are marked along each graph. Thus $u = 0.7$, corresponding to $\omega = 7000$ for the filter here discussed, is found near the point e' on graph 1, near G on graph 2, and near C on graphs 3 and 4. Graph 1 for the pure resistance goes off to infinity along the base line as ω approaches 4000, or as u approaches 0.4. It then returns along the parallel line $d'\ e'\ D$ as u approaches 1.0, then turns sharply through a positive quadrant and follows the straight line $D\ f$.

The load (2) containing the series condenser executes the closed oop $D\ F\ G\ H\ D$ and then pursues the curve $D\ J\ K$.

Loads (3) and (4) pursue somewhat similar curves which change suddenly at D. This point D at $0 + j\,\underline{1}$ is common to all four graphs at $u = 1$, or $\omega = \omega_0 = 10{,}000$. At the cut-off frequency,

therefore, all four loads subtend one circular quadrant when connected to this particular filter in one or more T sections.

Position Angle δ_A at the Sending End of the Filter Line.

In conformity with the regular rule affecting uniform artificial or real lines, operating at a single impressed frequency in the steady state, the position angle at the sending end A, as shown in Fig. 2, is

$$\delta_A = \Theta + \delta_B = n\theta + \theta' \qquad \text{hyps } \angle \quad (10)$$

δ_A clearly varies with the frequency ω, as do both the section angle θ and the load angle θ', according to (5) and (9). In Fig. 5, the broken line graph $a\,b\,c$ traces the vector position angle at A as u increases from 0 to 1.5.

Input and Output Currents.

If I_A is the input current at A (Fig. 2) in rms. amperes, and I_B the output current given to the load σ at B, in rms. amperes, we have[4] the well known relation

$$\frac{I_B}{I_A} = \frac{\cosh \delta_B}{\cosh \delta_A} \qquad \text{numeric } \angle \quad (11)$$

So that if the input current I_A is given, the output current I_B is obtained directly therefrom, with the cosine ratio of the two position angles δ_B and δ_A. The current strength at any section junction is likewise obtainable from the same rule.

Input and Output Impedances.

The impedance at and beyond the receiving end B in Fig. 2, or the output impedance is σ ohms $\angle$, a function of the frequency, unless the load σ is a pure resistance. The impedance at the input terminal A, or the input impedance is well known to be

$$Z_A = z_0 \tanh \delta_A \qquad \text{ohms } \angle \quad (12)$$

Both z_0 and δ_A vary with the impressed frequency f. If either the current I_A, or the emf. E_A is known at A, the other can be immediately found from Z_A.

[4] Bibliography, 1, 2, 6.

Input and Output Voltages.

The voltages at A and B being defined as E_A and E_B rms. volts $\angle$, the input and output voltages respectively, we have the well known relation

$$\frac{E_B}{E_A} = \frac{\sinh \mathfrak{d}_B}{\sinh \mathfrak{d}_A} \qquad \text{numeric } \angle \quad (13)$$

or the output voltage at B is the input voltage E_A times the sine ratio of corresponding position angles. The voltage at any intermediate junction can likewise be derived, for any given frequency, by the same rule.

Input and Output Power.

The vector output power P_B watts $\angle$ is obtainable directly from the vector output voltage E_B volts $\angle$, and the vector output current I_B amperes $\angle$. This vector power will be expressed either to voltage standard phase, or to current standard phase. In the former case, the voltage E_B is taken as of zero slope, and the current I_B is then taken with its slope in local reference thereto. Thus, taking the impressed input voltage E_A as of initial standard phase say $10 \angle 0°$ volts, the output voltage E_B might be say $8 \sphericalangle 600°$ volts, or 8 volts 600° in phase behind E_A; while the output current I_B might be say $0.1 \sphericalangle 540°$ amperes, or one-tenth ampere lagging 540° behind E_A. To find the output power at B, taking E_B as of local standard phase, we write

$$E_B = 8 \angle 0° \qquad \text{volts } \angle$$

and

$$I_B = 0.1 \angle 60° \qquad \text{amperes } \angle$$

because I_B leads E_B in phase by $600 - 540 = 60$ degrees. We then have

$$P_B = E_B \times I_B \qquad \text{watts } \angle \quad (14)$$

or in this case $0.8 \angle 60$ watts $= 0.4 + j\,0.693$ watts; i.e. 0.4 watt active, or delivered power, and 0.693 watt reactive, or circulating power, of the $+j$ type, which will represent condensive activity. If, on the other hand, we take the local current at B as the standard of phase, $E_B = 8 \sphericalangle 60°$ volts, and $I_B = 0.1 \angle 0°$ ampere, whence by (14), $P_B = 0.8 \sphericalangle 60°$ watt $= 0.4 - j\,0.693$. The active power is the same as before, but the reactive power is $-j$, or condensive.

At the sending-end, with E_A as phase standard,

$$P_A = E_A \times I_A \qquad \text{watts } \angle \quad (15)$$

Behavior of Filter under Surge-Impedance Load.

If in Fig. 2, the load σ be kept equal to z_0 as the frequency is changed, the load will behave like an indefinite prolongation of the same filter, or the line becomes infinitely long. The entering current at A, is then

$$I_A = \frac{E_A}{z_0} = E_A\, y_0 \qquad \text{amperes } \angle \qquad (16)$$

or if the impressed voltage is kept at 1 volt, at standard phase, over the entire range of frequency, the entering current will be vectorially equal to the surge admittance y_0. The voltage and current at B will then be

$$E_B = E_A \varepsilon^{-\Theta} \qquad \text{volts } \angle \qquad (17)$$

and

$$I_B = I_A \varepsilon^{-\Theta} \qquad \text{amperes } \angle \qquad (18)$$

From an inspection of Figs. (5) and (6), it will be seen that the entering current I_A will continuously increase as ω is varied from 0 to ω_0. At the critical frequency $f_0 = \omega_0/2\pi$, the entering current will be indefinitely great, or the filter acts like a short circuit, (assuming no internal losses in the filter); while above f_0, the current falls

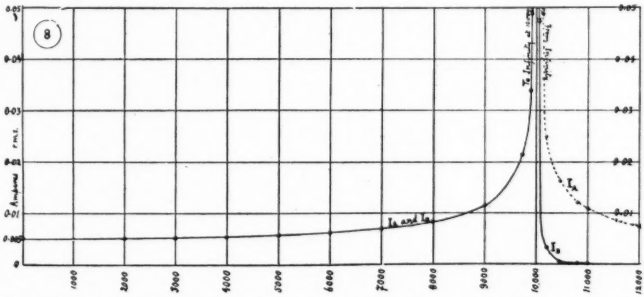

Fig. 8. Current strengths at terminals A and B of 5-Section filter Fig. 2 under surge-impedance load $\sigma = z_0$, as impressed frequency is varied from $\omega = 0$ to $\omega = 12000$.

off rapidly. This is shown in Fig. 8 by the curve I_A, commencing at 0.005 ampere with $\omega = 0$, and running off the sheet near $\omega = 9900$. At $\omega = 12000$, I_A has returned to 0.075 and is falling. The received current I_B is numerically equal to I_A up to the critical frequency. It then falls much faster than I_A and is very nearly 0 at $\omega = 11000$

Fig. 9 shows the corresponding vector diagram of I_B under varying impressed frequency ratio. At zero frequency, $I_B = PA = 5 \angle 0°$

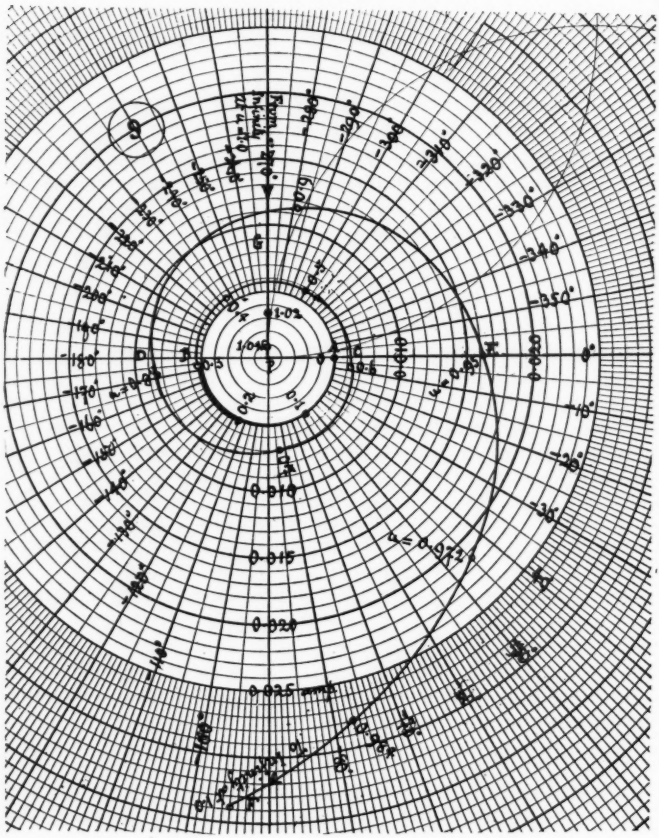

FIG. 9. Vector graph of current I_B under condition of Surge-Impedance Load.

milliamperes. At $u = 0.6$ or $\omega = 6000$, $I_B = OC$ or $6.25 \sphericalangle 368°$ milliamperes. The received current I_B is then more than one complete cycle behind the impressed emf. E_A. At $u = 0.9511$,

or $\omega = 9511$, $I_B = OE = 16 \sphericalangle 720°$ milliamperes. At OF, near $u = 0.984$, the received current is $32 \sphericalangle 810°$, and rapidly increasing. At $u = 1.0$ it becomes infinite, and above this critical value it returns rapidly along the line $G\,P$. At $u = 1.02$, it is $3.375 \sphericalangle 990°$ or 11 quadrants behind E_A in phase. The entering current I_A does not pursue this spiral path. As far as $u = 1$, it is numerically equal to I_B, but is in phase with E_A, or extends along the initial line $P\,A\,E$.

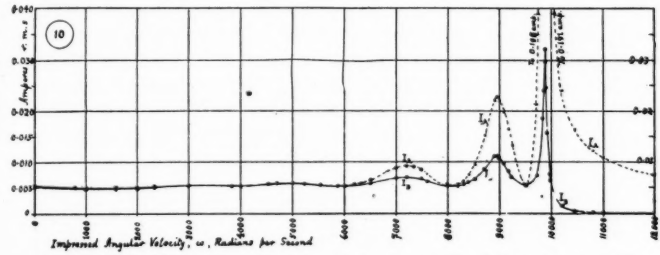

FIG. 10. Current strengths at terminals A and B of Five-Section Filter Fig. 2, under Constant-Resistance Load $\sigma = 183.3$ ohms, as the impressed angular velocity is varied from $\omega = 0$ to $\omega = 12000$.

Above $u = 1$, I_A returns from infinity along the line $F\,P$, opposite to $G\,P$. Before reaching the critical frequency, therefore, the phase difference between I_A and I_B increases from 0 to 10 quadrants or 900°. Beyond $u = 1$, it remains constant at 10 quadrants. I_B falls much more rapidly.

The expanding spiral executed by I_B in Fig. 9 is, up to $u = 1$,

$$I_B = \frac{y_{00}}{\sqrt{1 - u^2}} \sphericalangle (2n \sin^{-1} u) = y_{00} \sec \left(\frac{\beta}{2n}\right) \sphericalangle \beta$$

$$= y_{00} \sec \beta' \sphericalangle 2n\beta' \qquad \text{amperes} \angle \quad (19)$$

where $y_{00} = \sqrt{C/\mathcal{L}}$, the value of the surge admittance at zero frequency, in this case 0.005, n is the number of sections in the filter between A and B, in this case 5, and $u = \sin \beta$ and $\beta' = n\beta$. This type of expanding spiral may be called the *secant of the 2nth arc* spiral. If we consider a filter made up of a very large number of sections like Fig. 1, the vector current at the junctions 1, 2, 3, . . . in Fig. 2, will be $2n$th secant spirals, where $n = 1, 2, 3 \ldots$ successively. Such secant spirals characterize the vector currents at successive junctions along an infinitely long low-pass filter, of the types shown in Figs. 1 and 2.

BEHAVIOR OF THE FILTER UNDER THE CONSTANT PURE RESISTANCE LOAD $\sigma = 183.3$ OHMS.

Fig. 10 shows the corresponding current strengths I_A and I_B when the load σ at the receiving end is a constant resistance of 183.3 ohms, as ω is increased up to 12000. Instead of having one and only one resonant frequency as in Fig. 8, there are now five resonant frequencies; namely one at ω_0, no longer a short-circuit resonance, and lesser resonances at or near $\omega = 9000$, 7000, 5000 and 3000. There is very little difference between the magnitudes of I_A and I_B until the resonance near $\omega = 7000$ is reached. At this and subsequent resonances, I_A exceeds I_B. After reaching the frequency ratio $u = 0.9877$, when I_A goes to 0.191 ampere, and I_B to 0.032 ampere, both the currents diminish; but I_B falls much more rapidly. At the cut-off ratio $u = 1$, I_B has fallen to about 0.006 ampere.

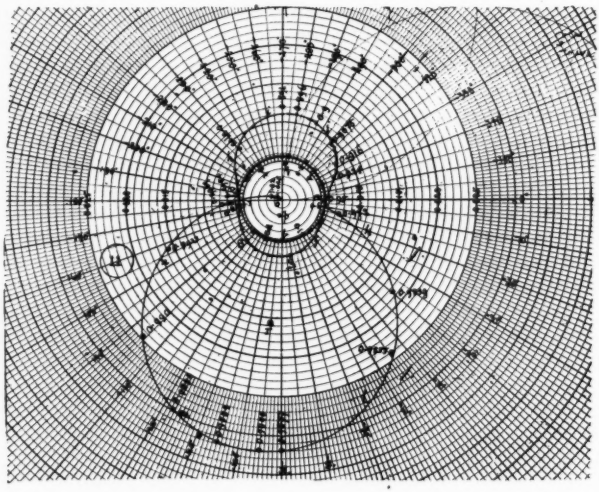

FIG. 11. Vector Graph of Current I_B under condition of Constant Resistance Load $\sigma = 183.3$ ohms, as the frequency ratio is increased from $u = 0$ to $u = 1.2$.

The corresponding vector graph of the current I_B is given in Fig. 11. At $u = 0$, this current is 5.4 milliamperes at standard phase, or along the initial line oa. At $u = 0.6$, the current has completed a nearly circular loop $a\ b\ c\ d$, and has reached the value $5.47 \sphericalangle 366°.9$, or

more than 4 quadrants behind the phase of E_A. It now executes the loop $f\ g\ h\ j$, and at $u = 0.95$, it has become $5.46 \angle 720°$ or 8 quadrants in lag. It now executes the nearly circular loop $k\ l\ m\ n$, with its diametral maximum near $u = 0.988$. At $u = 1.0$, it is about 6 milliamperes, 10 quadrants behind E_A (2 quadrants per section of filter).

The effect of substituting this constant resistance load for the surge-impedance load has been to change the vector graph of I_B from a secant $(\beta'/2n)$ spiral, such as that of Fig. 9, to a graph of a succession of approximately circular arcs, as in Fig. 11. The center of the arc $k\ l\ m\ n$ is at r, that of $h\ i\ j$ at q, and that of $f\ g$ at p. These centers lie nearly on one straight line, and fall successively on opposite sides of the origin O.

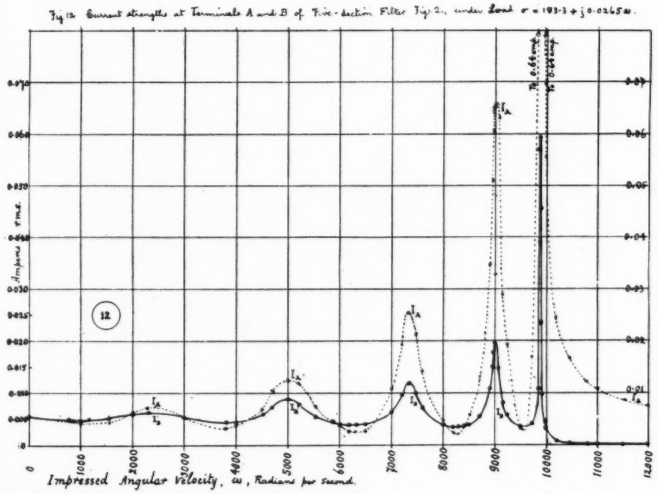

FIG. 12. Current strengths at Terminals A and B of Five-Section Filter Fig. 2, under Load $\sigma = 183.3 + j\,0.0265\ \omega$.

BEHAVIOR OF THE FILTER UNDER A CONSTANT REACTIVE LOAD $= 183.3 + j\,0.0265$ ohms $\angle$.

The effect of inserting a pure inductance of 26.5 millihenrys in circuit with 183.3 ohms resistance load, is shown in Fig. 12. The resonances produce in both I_A and I_B are now much more marked,

owing to the influence of the inductance in the load. There are five such resonances, one for each section of the filter. The principal resonance is at $\omega = 9886$, when I_A reaches 0.64 ampere, and I_B 0.059 ampere. The other resonances are near $\omega = 9000$, 7350, 5000 and 2400. At each resonance I_A exceeds I_B; but at intermediate frequencies I_A falls below I_B.

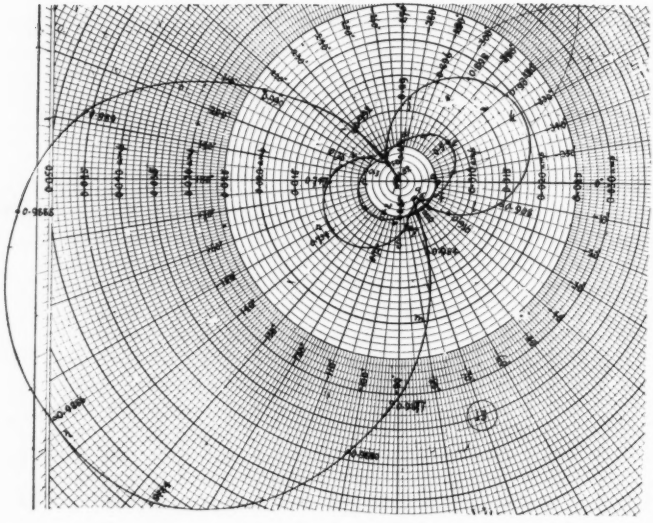

FIG. 13. Vector Graph of Current I_B under condition of a Reactive Load $\sigma = 183.3 + j\,0.0265\,\omega$, as the frequency ratio is increased from $u = 0$ to $u = 1.2$.

The corresponding vector graph of I_B appears in Fig. 13. ba is the initial current at $u = 0$. The first half loop $a\,b\,c$, goes as far as $u = 0.4$. The next $d\,e\,f$ reaches $u = 0.6$. The third $g\,h\,i$ goes to $u = 0.8$. The fourth $j\,k\,l$ reaches $u = 0.95$, and the last $m\,n\,o$, to $u = 1.02$ and beyond. These loops are only roughly circular, and are not nearly such good approximations to circles as those of Fig. 11. The centers of the loops in Fig. 11 lie nearly on a straight line inclined 85° with the reference axis; but in Fig. 13, the approximate centers lie more nearly on a line $n\,p\,k$, making an angle of 30° with the reference axis $p\,a$.

It may be noticed that between $u = 0.984$ and $u = 0.992$, the received current is very sensitive to change in impressed frequency, both in regard to magnitude and to phase, or the filter is a sensitive frequency indicator.

Table II indicates, for a few entries only, the method of computation employed, for the case of $= 183.3 + j\,0$ ohms, and Table III a few similar examples for the case of $= 183.3 + j\,0.0265\omega$.

Summary

(1) Any alternating-current filter, consisting of sections in series having the same surge impedance, forms an artificial electric line, and may advantageously be dealt with, in computation, by means of hyperbolic functions.

(2) The behavior of any such filter under surge-impedance load (virtually infinite line), is apt to be very different from its behavior under a constant load at the receiving end.

(3) In the case of a low-pass inductance-capacitance T-section filter with negligible losses, having a large number of sections, the vector current graphs at n successive junction points, as the impressed frequency is increased up to cut-off, are expanding secant spirals of the type $\rho = \rho_0 \sec (\beta/2n)$.

(4) Under the same conditions last named, but with a constant pure resistance load, the vector received current graph is a close approximation to an aggregation of resonant circles, one for each filter section.

(5) When the load in the last named case includes a constant inductance, the received current graph is a succession of resonant roughly circular loops, one for each filter section.

List of Symbols Employed.

β, β'	Circular angles of radius vector in a spiral (radians, quadrants or degrees).
C	Capacitance of a condenser in a filter (farads).
δ_A, δ_B, δ_P	Position angles of terminals A and B, or of a junction point P (hyps $\angle$).
E_A, E_B, E_P	Voltages at terminals A and B, or of a junction P (volts $\angle$).
f	Impressed frequency at the sending end (cyps).
f_0	Cut-off frequency (cyps).
$\theta = \theta_1 + j\theta_2$	Angle subtended by a filter section (hyps $\angle$).

TABLE II

EXAMPLES OF TABULAR COMPUTATION OF 5-SECTION FILTER IN FIG. 2, UNDER DIFFERENT SUCCESSIVE FREQUENCIES, WITH $1.0 \angle 0°$ VOLT IMPRESSED AT A, AND A LOAD OF 183.3 OHMS AT B.

u	$\frac{\theta}{2}$ hyps	θ hyps	Θ_q hyps	$\cos \frac{\theta}{2}$	z_0 ohms	$\frac{\sigma}{z_0}$	δ_B hyps	δ_A hyps	$\tanh \delta_A$	$Z_A = z_0 \tanh \delta_A$ ohms
0.2	$j\,0.2014$	$j\,0.4028$	$j\,1.282$	0.9797	195.9	0.935	1.70	$1.700 + j\,1.282$	$1.043 \angle 3°.$	$204.4 \angle 3°.0$
0.4	$j\,0.4115$	$j\,0.8230$	$j\,2.620$	0.9165	183.3	1	∞	∞	1	$183.3 \angle 0°$
0.6	$j\,0.6435$	$j\,1.287$	$j\,4.093$	0.800	160	1.145	$1.347 + j\,1$	$1.347 + j\,5.093$	$1.138 \angle 2°.35$	$182.2 \angle 2°.35$
0.8	$j\,0.9273$	$j\,1.8546$	$j\,5.905$	0.600	120	1.528	$0.783 + j\,1$	$0.783 + j\,6.905$	$1.490 \angle 7°.5$	$178.8 \angle 7°.5$
0.99	$j\,1.429$	$j\,2.858$	$j\,9.10$	0.141	28.2	6.50	$0.155 + j\,1$	$0.155 + j\,10.1$	$0.220 \angle 44°$	$6.20 \angle 44°$

$I_A = Y_A = 1/Z_A$ amp $\times 10^{-3}$	$\cosh \delta_A$	$\cosh \delta_B$	I_B amp $\times 10^{-3}$	$\sinh \delta_A$	$\sinh \delta_B$	E_B volts	$\sigma = E_B/I_B$ ohms.
$4.892 \angle 3°.0$	$2.675 \angle 117°$	$2.828 \angle 0°$	$5.172 \angle 114°$	$2.800 \angle 113°.8$	$2.646 \angle 0°$	$0.945 \angle 113°.8$	$182.7 \angle 0°.2$
$5.455 \angle 0°$	∞	∞	$5.455 \angle 235°.8$	∞	∞	$1 \angle 235°.8$	$183.3 \angle 0°$
$5.488 \angle 2°.35$	$1.80 \angle 99°.3$	$1.793 \angle 90°$	$5.467 \angle 6°.9$	$2.055 \angle 97°$	$2.053 \angle 90°$	$0.999 \angle 7°$	$182.7 \angle 0°.1$
$5.593 \angle 7°.5$	$0.88 \angle 257°$	$0.8655 \angle 90°$	$5.501 \angle 174°.5$	$1.315 \angle 264°.2$	$1.323 \angle 90°$	$1.006 \angle 174°.2$	$182.8 \angle 0°.3$
$161.2 \angle 44°$	$1.020 \angle 182°$	$1.0120 \angle 90°$	$24.6 \angle 136°$	$0.225 \angle 226°$	$0.1556 \angle 90°$	$4.498 \angle 136°$	$182.8 \angle 0°$

TABLE III.

EXAMPLES OF TABULAR VOLTAGES AND CURRENTS AT SUCCESSIVE TERMINALS OF FIVE-SECTION FILTER IN FIG. 2 WITH LOAD AT B OF $183.3 + j\,0.0265\,\omega$ OHMS, UNDER DIFFERENT IMPRESSED FREQUENCIES. $E_A = 1.0 \angle 0°$ VOLT.

u	ω	Z_A	I_A	E_1	I_1	E_2	I_2
		ohms	milliamp.	volts	milliamp.	volts	milliamp.
0.2	2000	$160.0 \angle 348°.2$	$6.25 \diagdown 348°.2$	$1.12 \diagdown 24°.9$	$5.69 \diagdown 8°.3$	$1.23 \diagdown 44°.9$	$5.10 \diagdown 34°$
0.4	4000	$281 \angle 339°.2$	$3.56 \diagdown 339°.2$	$0.958 \diagdown 26°$	$3.90 \diagdown 54°.5$	$0.624 \diagdown 76°$	$5.53 \diagdown 90°.2$
0.6	6000	$278 \angle 34°.3$	$3.6 \diagdown 34°.3$	$0.429 \diagdown 91°.3$	$6.78 \diagdown 83°.7$	$1.04 \diagdown 166°.8$	$3.04 \diagdown 143°.7$
0.8	8000	$214 \angle 56°.4$	$4.66 \diagdown 56°.4$	$0.786 \diagdown 157°.7$	$6.97 \diagdown 96°.4$	$0.61 \diagdown 195°.9$	$8.19 \diagdown 254°.7$
0.99	9900	$3.71 \angle 67°.8$	$268 \diagdown 67°.8$	$2.88 \diagdown 165°.3$	$250 \diagdown 247°.4$	$4.73 \diagdown 345°.2$	$210 \diagdown 424°.8$

E_3	I_3	E_4	I_4	E_B	I_B	σ
volts	milliamp.	volts	milliamp.	volts	milliamp.	ohms
$1.26 \diagdown 62°.4$	$4.85 \diagdown 63°.5$	$1.22 \diagdown 80°.4$	$5.46 \diagdown 92°.8$	$1.10 \diagdown 100°.9$	$5.85 \diagdown 117°$	$189.3 \angle 16°.1$
$0.750 \diagdown 160°.2$	$5.05 \diagdown 120°.8$	$1.03 \diagdown 195°$	$3.30 \diagdown 173°.2$	$0.904 \diagdown 214°.2$	$4.28 \diagdown 244°.3$	$211.8 \angle 30°.1$
$0.626 \diagdown 208°.3$	$6.25 \diagdown 249°.2$	$0.802 \diagdown 330°$	$5.17 \diagdown 283°.7$	$0.946 \diagdown 364°.3$	$3.89 \diagdown 405°.2$	$243 \angle 40°.9$
$1.08 \diagdown 349°.1$	$3.20 \diagdown 317°.6$	$0.398 \diagdown 450°.9$	$9.32 \diagdown 443°.4$	$1.05 \diagdown 537°.9$	$3.77 \diagdown 587°$	$280 \angle 49°.1$
$6 \diagdown 520°$	$157 \diagdown 603°.3$	$7 \diagdown 699°$	$93.2 \diagdown 783°.3$	$7 \diagdown 877°.8$	$23.1 \diagdown 932°.9$	$320.5 \angle 55°.1$

θ' Angle subtended by a load σ at receiving end (hyps $\angle$).

Θ Total angle subtended by a multiple-section filter (hyps $\angle$).

I_A, I_B, I_P Current strengths at terminals, or at a junction (amperes $\angle$).

$j=\sqrt{-1}$.

$\mathcal{L}$ Inductance in the series branches of a T-filter section (henrys).

n Number of sections in a multiple-section filter.

$\pi=3.14159\ldots$

P_A, P_B Power at terminals (watts $\angle$).

q Imaginary component of a complex angle expressed in circular quadrants.

ρ Radius vector of an expanding spiral.

ρ_0 Initial radius vector of an expanding spiral.

σ Impedance of a load at receiving end (ohms $\angle$).

u Frequency ratio f/f_0 (numeric).

$y_0 = 1/z_0$ Surge admittance of a T-filter at a given impressed frequency (ohms $\angle$).

$y_{00} = \sqrt{C/\mathcal{L}}$ Surge admittance of T-filter at zero frequency (ohms).

z_0 Surge impedance of T-filter at a given impressed frequency (ohms $\angle$).

$z_{00} = \sqrt{\mathcal{L}/C}$ Surge impedance of T-filter at zero frequency (ohms.)

$\omega = 2\pi f$ Impressed angular velocity (radians per second).

$\omega_0 = 2\pi f_0$ Cut-off angular velocity of filter (radians per second).

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